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Efficient approximate unitary designs from random Pauli rotations

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Outline:
1. Unitary t-design defn & spectral gap
2. Construction & results
3. Lie group $SU(N)$
4. Lie algebra $su(N)$
5. Proof: $su(2)$
6. Proof: $su(2^n)$
7. Discussion

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① An exact/approx. unitary t-design:

a distribution on $su(N)$ which matches exactly/approximately with the Haar distribution on all moments up to t

approximately → efficient Q. circuits

$$\mathcal{H}_{t,\nu} : \eta \longmapsto \mathbb{E}_{U\sim\nu} U^{\otimes t} \cdot \eta \cdot U^{\dagger,\otimes t}$$

t: the channel acts on t copies of the Hilbert space

ν: distribution on $SU(2^n)$

additive error $\|\mathcal{H}_{t,\mathrm{Haar}} - \mathcal{H}_{t,\nu}\|_\diamond \le \varepsilon$ — most known

multiplicative error $(1-\varepsilon)\,\mathcal{H}_{t,\mathrm{Haar}} \preccurlyeq \mathcal{H}_{t,\nu} \preccurlyeq (1+\varepsilon)\,\mathcal{H}_{t,\mathrm{Haar}}$ — stronger

↪ connect to robust quantum circuit complexity

spectral gap

vectorization of the channel

$$\Big\| \underbrace{\mathbb{E}_{U\sim \mathrm{Haar}} U^{\otimes t}\otimes \bar{U}^{\otimes t}}_{A} - \underbrace{\mathbb{E}_{U\sim\nu} U^{\otimes t}\otimes \bar{U}^{\otimes t}}_{B} \Big\|_\infty \le 1-\Delta$$

largest eval

why spectral gap?

Note $$(A-B)^2 = \underbrace{A^2}_{=A} + \underbrace{B^2}_{=\mathbb{E}_{U\sim\nu^{*2}} U^{\otimes t,t}} - \underbrace{2AB}_{=2A} = B^2 - A$$

$U = U_1 U_2 \quad U_1, U_2 \sim \nu$

↑ repeat twice

$$\Big\| \underbrace{\mathbb{E}_{U\sim \mathrm{Haar}} U^{\otimes t,t}}_{A} - \underbrace{\mathbb{E}_{U\sim\nu^{*k}} U^{\otimes t,t}}_{B^k} \Big\|_\infty \leq (1-\Delta)^k$$

Model : random walks on $SU(2^n)$

1/ ν defines one step of a random walk

2/ take k steps to mix

(k independent draws, then multiply k unitaries together)

spectral gap quantifies how fast the walk mixes

$$\#\text{steps} = O\Big(\underbrace{\frac{1}{\text{gap}}}_{=\frac{1}{\Delta}} \cdot (nt + \log(1/\varepsilon))\Big)$$

Spectral gap implies both additive & multiplicative designs

② Construction & results:

- Simple construction
- nice constants
- multiplicative error   bc we use a spectral gap argument
- any moment $t > 0$ all other t-design analysis is limited to a certain regime e.g. $t = O(2^n)$, $t = o(2^{n/\log n})$, $t = o(n^{1/2})$
- simple proof   as long as you know $su(2)$ rep theory

Each step sample $\exp(i\frac{\theta}{2}P)$ where $\theta \overset{\text{unif}}{\sim} [-\pi, \pi] \to$ can be discretized

$P \overset{\text{unif}}{\sim} P_n = \{1, X, Y, Z\}^{\otimes n} \setminus \{I^{\otimes n}\}$

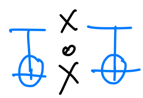
→ Q: Does anyone immediately knows how?

Assume all-to-all connection. $\exp(i\frac{\theta}{2}P)$ can be implemented w/ depth $O(\log n)$

e.g.

$$P = \begin{array}{ccc} & X & \\ & \otimes & \\ & 1 & \\ & \otimes & \\ S^\dagger & Y & S \\ & \otimes & \\ H & Z & H^\dagger \end{array} = \begin{array}{c} X \\ \otimes \\ 1 \\ \otimes \\ X \\ \otimes \\ X \end{array}$$

① use $\le 2n$ H & S to convert P into $\{1, X\}^{\otimes n}$ (depth 2)

②  $= \begin{array}{c} X \\ \otimes \\ 1 \end{array}$

CNOT of depth $O(\log n)$

<u>Theorem</u>: $\forall\, n, t \geq 1$

$$(*) = \Big\| \mathop{\mathbb{E}}_{\theta\sim(-\pi,\pi)} \mathop{\mathbb{E}}_{P\sim P_n} \exp(i\tfrac{\theta}{2}P)^{\otimes t} \otimes \exp(-i\tfrac{\theta}{2}\bar{P})^{\otimes t} - \mathop{\mathbb{E}}_{U\sim \text{Haar}} U^{\otimes t, t} \Big\|_\infty \leq 1 - \frac{1}{4t} - \frac{1}{4^n - 1}$$

By sampling k random Pauli rotations, i.e. $\exp(i\frac{\theta_k}{2}P_k) \cdots \exp(i\frac{\theta_1}{2}P_1)$

$(*) \leq 1 - \frac{1}{4t}$ directly implies

① ε-additive error t-designs if $k \geq 4t\left(\ln 2 \cdot nt + \log(\frac{1}{\varepsilon})\right)$

② ε-multiplicative   " $k \geq 4t\left(\ln 8 \cdot nt + \log(\frac{1}{\varepsilon})\right)$

Overall depth $= O\left(\log n \cdot t\, (nt + \log(\frac{1}{\varepsilon}))\right)$

Previous best spectral gap [Haf 22] $\Omega(t^{-4-o(1)})$

this work $\Omega(t^{-1})$

③ Lie group SU(N)

$\tau_t : U \mapsto U^{\otimes t} \otimes \bar{U}^{\otimes t}$ is a $SU(2^n)$ representation

call this tensor product representation

group rep : $\tau_t(U) \cdot \tau_t(V) = \tau_t(U \cdot V)$

τ_t is reducible $\Rightarrow$ decompose into irreps

Hence : $(*) = \max_{\rho \in \tau_t} \left\| \mathbb{E}_{\theta, P} \rho\left(\exp(i \frac{\theta}{2} P)\right) - \mathbb{E}_{U \sim \text{Haar}} \rho(U) \right\|_\infty$

ρ is a $SU(2^n)$ irreps that show up in τ_t

Prop : $\mathbb{E}_{U \sim \text{Haar}} \rho(U) = \begin{cases} 1 & \text{if } \rho \text{ is the trivial irrep} \\ 0 & \text{non-trivial} \end{cases}$

Pf. Schur's Lemma

Hence : $(*) = \max_{\substack{\rho \in \tau_t \\ \text{non-trivial}}} \left\| \mathbb{E}_{\theta, P} \rho\left(\exp(i \frac{\theta}{2} P)\right) \right\|_\infty$

which ρ occur in τ_t is well-understood

each ρ is labeled by a Young diagram

④ Lie algebra $su(N)$

algebra in the exponent of a unitary

$\exp(i\ H)$

$$\Rightarrow su(N) = \{ iH : H \text{ Hermitian} \}$$
$$= \mathbb{R}\text{-span} \{ iP : P \in \mathcal{P}_n \} \quad \text{if } N = 2^n$$

<u>Defn</u>: $J : su(N) \longrightarrow u(M)$ is an $su(N)$ representation

$\Downarrow$

$[J(A), J(B)] = J([A, B])$ commutator / Lie bracket

J linear map

Commutative diagram : $\forall$ $N \times N$ Hermitian matrix H

$$\begin{tikzcd} iH \in su(N) \arrow[d, "\exp"'] \\ \exp(iH) \in SU(N) \end{tikzcd}$$

$$\begin{tikzcd} iH \in su(N) \arrow[d, "\exp"'] & \\ \exp(iH) \in SU(N) \arrow[r, "\rho"'] & U(M) \end{tikzcd}$$

Lie group representation

induced Lie algebra representation

$$\begin{tikzcd} iH \in su(N) \arrow[r, "\rho_*"] \arrow[d, "\exp"'] & u(M) \arrow[d, "\exp"] \\ \exp(iH) \in SU(N) \arrow[r, "\rho"'] & U(M) \end{tikzcd}$$

Lie group representation

$\rho_* \Longleftrightarrow \rho$

1-to-1 correspondence

Key : $\forall$ $SU(N)$ representation ρ, $\forall$ $N \times N$ Hermitian H

$$\rho\left(\exp(iH)\right) = \exp\left(i \rho_*(H)\right)$$

e.g. recall $\tau_2(U) = (U \otimes \bar{U})^{\otimes 2}$

$$\tau_{2*}(H) = (H \otimes I - I \otimes \bar{H}) \otimes I \otimes I + I \otimes I \otimes (H \otimes I - I \otimes \bar{H})$$

$$\tau_{t*}(H) = \sum_{j=0}^{t-1} (I \otimes I)^{\otimes j} \otimes (H \otimes I - I \otimes \bar{H}) \otimes (I \otimes I)^{t-j-1}$$

Q : What are the eigenvalues of $\tau_{t*}(\frac{P}{2})$ $P \in P_n$?

Hence : $(*) = \max\limits_{\rho} \left\| \mathbb{E}_{\theta, P} \exp(i \rho_*(\frac{\theta}{2} P)) \right\|_\infty$

linear

$$= \max_{\rho} \left\| \mathbb{E}_{\theta, P} \exp(i \theta \rho_*(\tfrac{P}{2})) \right\|_\infty$$

Lemma 1 : eigenvalues of $\rho_*(\frac{P}{2})$ are integers in $[-t, t]$

ρ_* is a subrep of τ_{t*} + τ_{t*} only has integer evals in $[-t, t]$

Lemma 2 : eigenspectrum of $\rho_*(\frac{P}{2})$ is independent from $P \in P_n$

different Paulis are conjugated by a Clifford unitary

$$\int_{-\pi}^{\pi} e^{i\theta k} d\theta = \begin{cases} 0 & k \text{ non-zero integer} \\ 1 & k = 0 \end{cases}$$

$$= \max_{\rho} \left\| \mathbb{E}_{P} \operatorname{Ker}(\rho_*(\tfrac{P}{2})) \right\|_\infty$$

a projector onto the 0-eigenspace of $\rho_*(\frac{P}{2})$

$\forall$ non-zero Hermitian H

$$\ker H \precsim I - \frac{H^2}{\|H\|_\infty^2}$$

The only inequality !!

also $\forall P \in P_n$, $\|\rho_*(\frac{P}{2})\|_\infty \leq t$

Q: Anyone wants to guess what this is ?

Prop: $S = \sum\limits_{P \in P_n} \left[\rho_*(\frac{P}{2}) \right]^2 \propto I$ Casimir operator

Proof sketch : S commutes with $\rho_*(P)$ $\forall P \in P_n$ + Schur's Lemma

⑤ $n=1$ $su(2)$

$\forall$ irrep ρ of $su(2)$: $J_x = \rho_*(\frac{X}{2})$ $K_x = \mathrm{Ker}(\rho_*(\frac{X}{2}))$

$$\Big\| \mathop{\mathbb{E}}_{P \in P_1} K_P \Big\|_\infty = \frac{1}{3} \| K_x + K_y + K_z \|_\infty$$

$K_x \preceq I - \frac{J_x^2}{\ell^2}$

$$\leq \frac{1}{3} \Big\| 3I - \frac{(J_x^2 + J_y^2 + J_z^2)}{\ell^2} \Big\|_\infty$$

where $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

J_x has spectrum $\ell, \ell-1, \ell-2, \dots, -\ell$

$J_x^2 + J_y^2 + J_z^2 = \ell(\ell+1) \cdot I$

$$= 1 - \frac{1}{3} \cdot \frac{\ell^2 + \ell}{\ell^2} \leq \frac{2}{3}$$

Hence $(*) \leq \frac{2}{3}$

For $su(2)$, we can calculate $(*)$ i.e., the spectral gap exactly

Back to the general case:

$$(*) = \max_\rho \Big\| \mathop{\mathbb{E}}_{P \in P_n} K_P \Big\|_\infty \leq 1 - \min_\rho \Big\| \frac{\mathop{\mathbb{E}}_{P \in P_n} J_P^2}{\ell^2} \Big\|_\infty$$

$\mathrm{Ker}\, H \preceq I - \frac{H^2}{\|H\|_\infty^2}$

$\|J_P\|_\infty = \ell \leq t$

$\ell\infty$-norm is kept because

$\sum_P J_P^2 \propto I$

Goal: Prove $\Big\| \mathop{\mathbb{E}}_{P \in P_n} [\rho_*(\frac{P}{2})]^2 \Big\|_\infty \geq \frac{\ell}{4}$ $\forall \rho \in \mathcal{T}_t$ non-trivial

$\Downarrow$

$(*) \leq 1 - \frac{1}{4t}$

⑥ $su(2^n)$:

$\forall$ irrep ρ of $su(2^n)$, $\mathbb{E}_{P\in P_n}[\rho_*(\frac{P}{2})]^2$ has explicit formula

Nevertheless, let us see a simple counting argument from $su(2)$

Suppose $\quad J_{Z_1}|v\rangle = \ell|v\rangle \qquad \ell = \|J_{Z_1}\|_\infty$

$\Rightarrow \langle v|J_{Z_1}^2|v\rangle = \ell^2$

Consider all $Q, W \in P_n$ s.t. $\{Q, W, Z_1\}$ forms $su(2)$-subalgebra

e.g. $Z I^{\otimes n}$, $X \underline{\ \cdot\cdot\ }$, $Y \underline{\ \cdot\cdot\ }$

There are 4^{n-1} such $su(2)$-subalgebras

For each triple

$\langle v|J_{Z_1}^2 + J_Q^2 + J_W^2|v\rangle = \ell(\ell+1)$

Hence, $\langle v|J_Q^2 + J_W^2|v\rangle = \ell$

Hence, $\langle v|\sum_{P\in P_n} J_P^2|v\rangle \geq \underset{\underset{\langle v|J_{Z_1}^2|v\rangle}{\shortparallel}}{\ell^2} + 4^{n-1}\cdot \ell$

$\Rightarrow \|\mathbb{E}_{P\in P_n} J_P^2\|_\infty \geq \frac{1}{4}\ell + \frac{1}{4^n-1}\cdot\ell^2$